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ON GRADED 2-ABSORBING AND WEAKLY GRADED 2-ABSORBING SUBMODULES

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Abstract

Let G be a group with identity e. Let R be a G-graded commutative ring and M be a graded R-module. In this paper, we introduce the concepts of graded 2-absorbing and weakly graded 2-absorbing submodules and study some of their properties.

1. Introduction

Graded prime and graded primary ideals of a commutative graded ring R with a nonzero identity have been introduced and studied by Refai and Al-Zoubi in [8]. Graded prime submodules of a graded R-module have been studied by Ebrahimi Atani and Farzalipour in [1, 5]. Also, graded weakly prime submodules of graded R-modules have been studied in [2].

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Here, we study graded 2-absorbing and weakly graded 2-absorbing submodules of graded modules over graded commutative rings. Before we state some results, let us introduce some notations and terminologies. Let G be a group with identity e and R be a commutative ring. Then R is a G-graded ring if there exist additive subgroups R_g of R such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. We denote this by (R, G). The elements of R_g are called homogeneous of degree g, where R_g are additive subgroups of R indexed by the elements $g \in G$. If $x \in R$, then x can be written uniquely as $\sum_{g \in G} x_g$, where x_g is the component of x in R_g. Also, we write $h(R) = \bigcup_{g \in G} R_g$. Moreover, R_e is a subring of R and $1_R \in R_e$. Let I be an ideal of R. Then I is called graded ideal of (R, G), if $I = \bigoplus_{g \in G} (I \cap R_g)$. Thus, if $x \in I$, then $x = \sum_{g \in G} x_g$ with $x_g \in I$. An ideal of a G-graded ring need not be G-graded. Let $R = \bigoplus_{g \in G} R_g$ be a G-graded ring and let I be a graded ideal of R. Then the quotient ring R/I is also a G-graded ring. Indeed, $R/I = \bigoplus_{g \in G} (R/I)_g$, where $(R/I)_g = \{x + I : x \in R_g\}$. For simplicity, we will denote the graded ring (R, G) by R, see [6].

Let R be a G-graded ring and M be an R-module. We say that M is a G-graded R-module (or graded R-module), if there exists a family of subgroups $\{M_g\}_{g \in G}$ of M such that $M = \bigoplus_{g \in G} M_g$ (as abelian groups) and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. Here, $R_g M_h$ denotes the additive subgroup of M consisting of all finite sums of elements $r_g s_h$ with $r_g \in R_g$ and $s_h \in M_h$. Also, we write $h(M) = \bigcup_{g \in G} M_g$ and the elements of $h(M)$ are called homogeneous. Let $M = \bigoplus_{g \in G} M_g$ be a graded R-module and N be a submodule of M. Then N is called a graded submodule of M, if

$N = \underset{g \in G}{\oplus} N_g$, where $N_g = N \cap M_g$ for $g \in G$. In this case, N_g is called the g-component of N. Moreover, M/N becomes a G-graded R-module with g-component $(M/N)_g = (M_g + N)/N$ for $g \in G$, see [6].

Let R be a G-graded ring and $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then the ring of fraction $S^{-1}R$ is a graded ring, which is called graded ring of fractions. Indeed, $S^{-1}R = \underset{g \in G}{\oplus}(S^{-1}R)_g$, where $(S^{-1}R)_g = \{r/s : r \in R, s \in S \text{ and } g = (\deg s)^{-1}(\deg r)\}$. Let M be a graded module over a G-graded ring R and $S \subseteq h(R)$ be a multiplicatively closed subset of R. The module of fraction $S^{-1}M$ over a graded ring $S^{-1}R$ is a graded module, which is called module of fractions, if $S^{-1}M = \underset{g \in G}{\oplus}(S^{-1}M)_g$, where $(S^{-1}M)_g = \{m/s : m \in M, s \in S$ and $g = (\deg s)^{-1}(\deg m)\}$. We write $h(S^{-1}R) = \underset{g \in G}{\cup}(S^{-1}R)_g$ and $h(S^{-1}M) = \underset{g \in G}{\cup}(S^{-1}M)_g$.

A graded R-module M is called graded cyclic if $M = Rm$, where $m \in h(M)$. A proper graded ideal I of R is said to be graded 2-absorbing (resp., weakly graded 2-absorbing) ideal if whenever $r, s, t \in h(R)$ with $rst \in I$ (resp., $0 \neq rst \in I$), then either $rs \in I$ or $rt \in I$ or $st \in I$. A proper graded ideal P of R is said to be graded prime (resp., graded weakly prime) ideal if whenever $r, s \in h(R)$ with $rs \in P$ (resp., $0 \neq rs \in P$), then either $r \in P$ or $s \in P$, see [3, 8]. A proper graded submodule N of a graded module M is said to be graded prime (resp., graded weakly prime) submodule if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in N$ (resp., $0 \neq rm \in N$), then either $r \in (N :_R M)$ or $m \in N$, see [1, 2, 5, 7].

2. Graded 2-Absorbing Submodules

In this section, we define the graded 2-absorbing submodules and give some of their basic properties.

Lemma 2.1 ([1] and [4]). *Let R be a G-graded ring, M be a graded R-module, and N be a graded submodule of M. Then the following hold*:

(1) $(N :_R M) = \{r \in R : rM \subseteq N\}$ *is a graded ideal of R.*

(2) *rN and Rm are graded submodules of M, where $r \in h(R)$ and $m \in h(M)$.*

Definition 2.2. Let R be a G-graded ring, M be a graded R-module, N be a graded submodule of M and let $g \in G$.

(i) We say that N_g is g-2-absorbing submodule of R_e-module M_g, if $N_g \neq M_g$; and whenever $r, s \in R_e$ and $m \in M_g$ with $rsm \in N_g$, then either $rs \in (N_g :_{R_e} M_g)$ or $rm \in N_g$ or $sm \in N_g$.

(ii) We say that N is a graded 2-absorbing submodule of M, if $N \neq M$; and whenever $r, s \in h(R)$ and $m \in h(M)$ with $rsm \in N$, then either $rs \in (N :_R M)$ or $rm \in N$ or $sm \in N$.

Lemma 2.3. *Let R be a G-graded ring, M be a graded R-module, and N be a graded submodule of M. If N is a graded 2-absorbing submodule of M, then N_g is a g-2-absorbing R_e-submodule of M_g for every $g \in G$.*

Proof. Suppose that N is a graded 2-absorbing submodule of M. For $g \in G$, assume that $rsm \in N_g \subseteq N$, where $r, s \in R_e$ and $m \in M_g$. Since N is a graded 2-absorbing submodule of M, we have either $rs \in (N :_R M)$ or $rm \in N$ or $sm \in N$. Since $M_g \subseteq M$ and $N_g = N \cap M_g$, we conclude that either $rs \in (N_g :_{R_e} M_g)$ or $rm \in N_g$ or $sm \in N_g$. So N_g is g-2-absorbing R_e-submodule of M_g.

□

Theorem 2.4. *Let R be a G-graded ring, M be a graded R-module, and N be a graded submodule of M. If N is an intersection of two graded prime submodules of M, then N is a graded 2-absorbing submodule of M.*

Proof. Suppose that $N = N_1 \cap N_2$, where N_1 and N_2 are graded prime submodules of M. Let $r, s \in h(R)$ and $m \in h(M)$ with $rsm \in N$. Since $rsm \in N_1$ and N_1 is a graded prime submodule of M, we have either $r \in (N_1 :_R M)$ or $s \in (N_1 :_R M)$ or $m \in N_1$. Since $rsm \in N_2$ and N_2 is a graded prime submodule of M, we have either $r \in (N_2 :_R M)$ or $s \in (N_2 :_R M)$ or $m \in N_2$. If $r \in (N_1 :_R M)$ and $s \in (N_2 :_R M)$, then $rs \in (N :_R M)$. If $r \in (N_1 :_R M)$ and $r \in (N_2 :_R M)$, then $r \in (N_1 :_R M) \cap (N_2 :_R M) = (N_1 \cap N_2 :_R M)$, and hence $rs \in (N :_R M)$. If $r \in (N_1 :_R M)$ and $m \in M$, then $rm \in N$. In other cases, we do the same. Thus N is a graded 2-absorbing submodule of M.

$\square$

Theorem 2.5. *Let R be a G-graded ring, M be a graded R-module, and N be a graded submodule of M. Let $g \in G$ such that N_g is a g-2-absorbing R_e-submodule of M_g. Then the following hold:*

(i) *For every R_e-submodule V of M_g and every pair of elements $r, s \in R_e$ such that $rsV \subseteq N_g$, either $rs \in (N_g :_{R_e} M_g)$ or $rV \subseteq N_g$ or $sV \subseteq N_g$.*

(ii) *$(N_g :_{R_e} M_g)$ is a g-2-absorbing ideal of R_e.*

Proof. (i) Suppose that $r, s \in R_e, V$ is R_e-submodule of M_g, $rsV \subseteq N_g$, $rV \not\subseteq N_g$, and $sV \not\subseteq N_g$. We show that $rs \in (N_g :_{R_e} M_g)$. There are $v_1, v_2 \in V$ such that $rv_1 \notin N_g$ and $sv_2 \notin N_g$. Since $rs(v_1 +$

$v_2) \in N_g$ and N_g is g-2-absorbing R_e-submodule of M_g, we conclude that either $rs \in (N_g :_{R_e} M_g)$ or $s(v_1 + v_2) \in N_g$ or $r(v_1 + v_2) \in N_g$. If $s(v_1 + v_2) \in N_g$, then $sv_1 \notin N_g$, since $sv_1 \in N_g$ implies that $sv_2 \in N_g$, which is a contradiction. Now, since $rsv_1 \in N_g$, $rv_1 \notin N_g$ and $sv_1 \notin N_g$, we have $rs \in (N_g :_{R_e} M_g)$. With a same argument, we can show that if $r(v_1 + v_2) \in N_g$, then $rs \in (N_g :_{R_e} M_g)$.

(ii) As N_g is a g-2-absorbing, we get $(N_g :_{R_e} M_g) \neq R_e$. Now suppose that $rst \in (N_g :_{R_e} M_g)$, for some $r, s, t \in R_e$. Let $V = tM_g$, then V is an R_e-submodule of M_g. Since N_g is a g-2-absorbing submodule of M_g and $rsV \subseteq N_g$, it follows from (i) that either $rs \in (N_g :_{R_e} M_g)$ or $rt \subseteq (N_g :_{R_e} M_g)$ or $st \subseteq (N_g :_{R_e} M_g)$. Therefore, $(N_g :_{R_e} M_g)$ is a g-2-absorbing ideal of R_e.

□

Theorem 2.6. *Let R be a G-graded ring, M be a graded R-module, and N be a graded submodule of M. Let $g \in G$ such that M_g is cyclic R_e-submodule. Then N_g is g-2-absorbing R_e-submodule of M_g, if and only if $(N_g :_{R_e} M_g)$ is a g-2-absorbing ideal of R_e.*

Proof. Suppose that N_g is a g-2-absorbing submodule of M_g. By Theorem 2.5 (ii), we have $(N_g :_{R_e} M_g)$ is a g-2-absorbing ideal of R_e. Conversely, assume that $(N_g :_{R_e} M_g)$ is a g-2-absorbing ideal of R_e and let $rsm \in N_g$, for some $r, s \in R_e$ and for some $m \in M_g$. Since M_g is a cyclic R_e-module, $M_g = R_e x$ for some $x \in M_g$. Hence, there exists $t \in R_e$ with $m = tx$. Hence $rst \in (N_g :_{R_e} x) = (N_g :_{R_e} M_g)$. Since $(N_g :_{R_e} M_g)$ is g-2-absorbing ideal of R_e and $rst \in (N_g :_{R_e} M_g)$, we conclude that either $rs \in (N_g :_{R_e} M_g)$ or $rt \subseteq (N_g :_{R_e} M_g)$ or $st \subseteq (N_g :_{R_e} M_g)$. Thus $rs \in (N_g :_{R_e} M_g)$ or $rm \in N_g$ or $sm \in N_g$. Therefore, N_g is a g-2-absorbing submodule of M_g.

□

Theorem 2.7. *Let R be a G-graded ring, M be a graded R-module, N and V be graded R-submodules of M with* $V \subseteq N$. *Then N is a graded 2-absorbing submodule of M, if and only if* N/V *is a graded 2-absorbing R-submodule of* M/V.

Proof. Suppose that N is a graded 2-absorbing submodule of M. Let $r, s \in h(R)$, $m \in h(M)$, and $rs(m + V) \in N/V$. Since N is a graded 2-absorbing R-submodule of M and $rsm \in N$, we have either $rs \in (N :_R M)$ or $rm \in N$ or $sm \in N$. Hence either $rs \in (N/V :_R M/V)$ or $r(m + V) \in N/V$ or $s(m + V) \in N/V$. Therefore, N/V is a graded 2-absorbing R-submodule of M/V. Conversely, suppose that N/V is a graded 2-absorbing R-submodule of M/V. Let $rsm \in N$, for some $r, s \in h(R)$ and for some $m \in h(M)$. Since N/V is a graded 2-absorbing R-submodule of M/V and $rs(m + V) \in N/V$, we conclude that either $rs \in (N/V :_R M/V)$ or $r(m + V) \in N/V$ or $s(m + V) \in N/V$ and hence either $rs \in (N :_R M)$ or $rm \in N$ or $sm \in N$. Therefore, N is a graded 2-absorbing R-submodule of M.

□

Recall that the graded radical of a graded ideal I, denoted by $Gr(I)$, is the set of all $x \in R$ such that for each $g \in G$, there exists $n_g > 0$ with $x_g^{n_g} \in I$. Note that, if r is a homogeneous element, then $r \in Gr(I)$, if and only if $r^n \in I$ for some $n \in \mathbb{N}$, see [8]. A proper graded submodule N of a graded module M is said to be graded primary submodule, if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in N$, then either $m \in N$ or $r \in Gr((N :_R M))$, see [7]. The following theorem shows the relationship between graded primary submodules and graded 2-absorbing submodules:

Theorem 2.8. *Let R be a G-graded ring, M be a graded R-module, and N be a graded submodule of M. If N is a graded primary submodule of M and* $R/(N :_R M)$ *has no nonzero nilpotent elements, then N is a graded* 2*-absorbing submodule of M.*

Proof. Suppose that $rsm \in N, rm \notin N$ and $sm \notin N$ for some $r, s \in h(R)$ and for some $m \in h(M)$. We show that $rs \in (N :_R M)$. Since N is a graded primary submodule of $M, rsm \in N, rm \notin N$ and $sm \notin N$, we have $r, s \in Gr((N :_R M))$. Thus $(rs)^k \in (N :_R M)$ for some positive integer k. Since $R/(N :_R M)$ has no nonzero nilpotent element, we have $rs \in (N :_R M)$. Therefore, N is a graded 2-absorbing submodule of M.

□

A graded zero-divisor on a graded R-module M is an element $r \in h(R)$ for which there exists $m \in h(M)$ such that $m \neq 0$ but $rm = 0$. The set of all graded zero-divisors on M is denoted by $G - Zdv_R(M)$. The following result studies the behaviour of graded 2-absorbing submodules under localization.

Theorem 2.9. *Let R be a G-graded ring, M be a graded R-module, and* $S \subseteq h(R)$ *be a multiplication closed subset of R. Then the following hold*:

(1) *If N is a graded* 2*-absorbing submodule of M, then* $S^{-1}N$ *is a graded* 2*-absorbing submodule of* $S^{-1}M$.

(2) *If* $S^{-1}N$ *is a graded* 2*-absorbing submodule of* $S^{-1}M$ *and* $S \cap G - Zdv_R(M/N) = \phi$, *then N is a graded* 2*-absorbing submodule of M.*

Proof. (1) Suppose that $\frac{r_1}{s}\frac{r_2}{t}\frac{m}{l} \in S^{-1}N$ for some $\frac{r_1}{s}, \frac{r_2}{t} \in h(S^{-1}R)$ and for some $\frac{m}{l} \in h(S^{-1}M)$. Hence, there exists $k \in S$ such that $kr_1r_2m \in N$. Since N is a graded 2-absorbing submodule of M and $kr_1r_2m \in N$, we conclude that either $kr_1m \in N$ or $r_2m \in N$ or $kr_1r_2 \in (N :_R M)$. If $kr_1m \in N$, then $\frac{kr_1m}{ksl} = \frac{r_1}{s}\frac{m}{l} \in S^{-1}N$. If $r_2m \in N$, then $\frac{r_2m}{tl} = \frac{r_2}{t}\frac{m}{l} \in S^{-1}N$. So assume that $kr_1r_2 \in (N :_R M)$. Hence $\frac{kr_1r_2}{kst}$ $= \frac{r_1r_2}{st} \in S^{-1}(N :_R M)$. Thus $\frac{r_1}{s}\frac{r_2}{t} \in (S^{-1}N :_{S^{-1}R} S^{-1}M)$. Therefore, $S^{-1}N$ is a graded 2-absorbing submodule of $S^{-1}M$.

(2) Suppose that $rtm \in N$ for some $r, t \in h(R)$ and for some $m \in h(M)$. Hence $\frac{rtm}{1} = \frac{r}{1}\frac{t}{1}\frac{m}{1} \in S^{-1}N$. Since $S^{-1}N$ is a graded 2-absorbing submodule of $S^{-1}M$, we conclude that either $\frac{r}{1}\frac{m}{1} = \frac{rm}{1} \in$ $S^{-1}N$ or $\frac{t}{1}\frac{m}{1} = \frac{tm}{1} \in S^{-1}N$ or $\frac{r}{1}\frac{t}{1} = \frac{rt}{1} \in (S^{-1}N :_{S^{-1}R} S^{-1}M)$. If $\frac{rm}{1}$ $\in S^{-1}N$, then there exists $s \in S$ such that $srm \in N$ and since $S \cap G - Zdv_R(M/N) = \phi$, we have $rm \in N$. With a same argument, we can show that if $\frac{tm}{1} \in S^{-1}N$, then $tm \in N$. So assume that $\frac{rt}{1} \in (S^{-1}N :_{S^{-1}R} S^{-1}M)$ and hence $rtS^{-1}M \subseteq S^{-1}N$. Now, we have to show that $rt \in (N :_R M)$. Let $l \in h(M)$, hence $\frac{rtl}{1} \in rtS^{-1}M \subseteq S^{-1}N$. So that there exists $a \in S$ such that $artl \in N$. since $S \cap G - Zdv_R(M / N) = \phi$, we have $rtl \in N$. Thus $rt \in (N :_R M)$. Therefore, N is a graded 2-absorbing submodule of M. □

Theorem 2.10. *Let R be a G-graded ring, M be a graded R-module, N be a graded submodule of M and $g \in G$. If N_g is a g-2-absorbing R_e-submodule of M_g, then $(N_g :_{R_e} m)$ is a g-2-absorbing ideal of R_e, for all $m \in M_g$.*

Proof. Let $m \in M_g$. Suppose that $rst \in (N_g :_{R_e} m)$, $rs \notin (N_g :_{R_e} m)$ and $rt \notin (N_g :_{R_e} m)$, for some $r, s, t \in R_e$. We show that $st \in (N_g :_{R_e} m)$. Since N_g is a g-2-absorbing R_e-submodule of M_g, $rstm \in N_g$, $rsm \notin N_g$ and $rtm \notin N_g$, we conclude that $rst \in (N_g :_{R_e} M_g)$. By Theorem 2.5 (ii), we have $(N_g :_{R_e} M_g)$ is a g-2-absorbing ideal of R_e. Since $rst \in (N_g :_{R_e} M_g)$, $rs \notin (N_g :_{R_e} M_g)$ and $rt \notin (N_g :_{R_e} M_g)$, we have $st \in (N_g :_{R_e} M_g) \subseteq (N_g :_{R_e} m)$. Therefore, $(N_g :_{R_e} m)$ is a g-2-absorbing ideal of R_e.

□

3. Weakly Graded 2-Absorbing Submodules

In this section, we define the weakly graded 2-absorbing submodules and give some of their basic properties.

Definition 3.1. Let R be a G-graded ring, M be a graded R-module, N be a graded submodule of M and let $g \in G$.

(i) We say that N_g is a weakly g-2-absorbing submodule of R_e-module M_g, if $N_g \neq M_g$; and whenever $r, s \in R_e$ and $m \in M_g$ with $0 \neq rsm \in N_g$, then either $rs \in (N_g :_{R_e} M_g)$ or $rm \in N_g$ or $sm \in N_g$.

(ii) We say that N is a weakly graded 2-absorbing submodule of M, if $N \neq M$; and whenever $r, s \in h(R)$ and $m \in h(M)$ with $0 \neq rsm \in N$, then either $rs \in (N :_R M)$ or $rm \in N$ or $sm \in N$.

Clearly, a graded 2-absorbing submodule of M (resp., a g-2-absorbing R_e-submodule of M_g) is a weakly graded 2-absorbing submodule of M (resp., weakly g-2-absorbing R_e-submodule of M_g). However, since $\{0\}$ is always a weakly graded 2-absorbing submodule of M (resp., a weakly g-2-absorbing R_e-submodule of M_g) (by definition), a weakly graded 2-absorbing submodule of M (resp., a weakly g-2-absorbing R_e-submodule of M_g) need not be graded 2-absorbing (resp., g-2-absorbing).

Lemma 3.2. *Let R be a G-graded ring, M be a graded R-module, and N be a graded submodule of M. If N is a weakly graded 2-absorbing submodule of M, then N_g is a weakly g-2-absorbing R_e-submodule of M_g for every $g \in G$.*

Proof. The proof is similar to that of Lemma 2.3.

□

Theorem 3.3. *Let R be a G-graded ring, M be a faithful graded cyclic R-module, and N be a graded submodule of M. Then N is a weakly graded 2-absorbing submodule of M, if and only if $(N :_R M)$ is a weakly graded 2-absorbing ideal of R.*

Proof. Suppose that $M = Rm$ for some $m \in h(M)$. Assume first that N is a weakly graded 2-absorbing submodule of M. Let $r, s, t \in h(R)$, $0 \neq rst \in (N :_R M)$, $rt \notin (N :_R M)$ and $st \notin (N :_R M)$. We show that $rs \in (N :_R M)$. Then, there are $x_1, x_2 \in h(R)$ such that $rtx_1m \notin N$ and $stx_2m \notin N$ and hence $rtm \notin N$ and $stm \notin N$. Since M is faithful and $0 \neq rst$, we have $rstM \neq \{0\}$ and hence $rstm \neq 0$. Since N is a weakly graded 2-absorbing submodule of M, $0 \neq rstm \in N$, $rtm \notin N$ and $stm \notin N$, we conclude that $rs \in (N :_R M)$. Conversely, assume that $(N :_R M)$ is a weakly graded 2-absorbing ideal of R. Let $0 \neq rsm' \in N$ for some $r, s \in h(M)$ and for some $m' \in h(M)$. Then, there exists $t \in h(R)$ with $m' = tm$. Thus $0 \neq rst \in (N :_R m) = (N :_R M)$. Since $(N :_R M)$ is a weakly graded 2-absorbing ideal of R, we have either $rs \in (N :_R M)$ or $rt \in (N :_R M)$ or $st \in (N :_R M)$, and hence either $rs \in (N :_R M)$ or $rtm = rm' \in N$ or $stm = sm' \in N$. Therefore, N is a weakly graded 2-absorbing submodule of M. □

Theorem 3.4. *Let R be a G-graded ring, M be a graded R-module, $x \in M_g$ and $r \in R_e$. Then the following hold*:

(i) *If $ann_{M_g}(r) \subseteq rM_g$, then the R_e-submodule rM_g is a weakly g-2-absorbing submodule of M_g, if and only if it is a g-2-absorbing submodule of M_g.*

(ii) *If $ann_{R_e}(x) \subseteq (R_e x :_{R_e} M_g)$, then the R_e-submodule $R_e x$ is a weakly g-2-absorbing submodule of M_g, if and only if it is a g-2-absorbing submodule of M_g.*

Proof. (i) Suppose that rM_g is a weakly g-2-absorbing submodule of M_g. Let $a, b, \in R_e$ and $m \in M_g$ with $abm \in rM_g$. If $0 \neq abm$, then either $ab \in (rM_g :_{R_e} M_g)$ or $am \in rM_g$ or $bm \in rM_g$, since rM_g is a weakly g-2-absorbing R_e-submodule of M_g. So assume that $abm = 0$. Hence $a(b + r)m = arm \in rM_g$. Assume first that $0 \neq a(b + r)m$. Since rM_g is a weakly g-2-absorbing R_e-submodule of M_g and $0 \neq a(b + r)m \in rM_g$, we have either $a(b + r) \in (rM_g :_{R_e} M_g)$ or $am \in rM_g$ or $(b + r)m \in rM_g$, and since $r \in (rM_g :_{R_e} M_g)$, we conclude that either

$ab \in (rM_g :_{R_e} M_g)$ or $am \in rM_g$ or $bm \in rM_g$. Hence, we may assume that $a(b + r)m = 0$ and thus $arm = 0$. Hence $am \in ann_{M_g}(r) \subseteq rM_g$. Therefore, rM_g is a g-2-absorbing submodule of M_g. The converse is obvious.

(ii) The proof is similar to that of part (i).

□

The following results provides some conditions under which a weakly g-2-absorbing R_e-submodule of M_g is a g-2-absorbing. First, we need the following lemma:

Lemma 3.5. *Let R be a G-graded ring, M be a graded R-module, and N be a graded submodule of M. Let $g \in G$ such that N_g is weakly g-2-absorbing R_e-submodule of M_g and let $a, b \in R_e$, $m \in M_g$ such that $abm \in N_g$, $am, bm \notin N_g$ and $ab \notin (N_g :_{R_e} M_g)$. Then the following hold*:

(i) $abm = 0$.

(ii) $abN_g = \{0\}$.

(iii) $am(N_g :_{R_e} M_g) = bm(N_g :_{R_e} M_g) = \{0\}$.

(iv) $(N_g :_{R_e} M_g)^2 m = \{0\}$.

(v) $a(N_g :_{R_e} M_g)N_g = b(N_g :_{R_e} M_g)N_g = \{0\}$.

Proof. (i) Since N_g is a weakly g-2-absorbing R_e-submodule of M_g, $am, bm \notin N_g$ and $ab \notin (N_g :_{R_e} M_g)$, we have $abm = 0$.

(ii) Assume that $abN_g \neq \{0\}$, then there exists an $n \in N_g$ such that $abn \neq 0$. Hence $0 \neq abn = ab(m + n) \in N_g$. Since N_g is a weakly g-2-absorbing R_e-submodule of M_g, we have either $ab \in (N_g :_{R_e} M_g)$

or $a(m+n) \in N_g$ or $b(m+n) \in N_g$. Hence either $ab \in (N_g :_{R_e} M_g)$ or $am \in N_g$ or $bm \in N_g$, which is impossible. Consequently, $abN_g = \{0\}$.

(iii) Assume that $am(N_g :_{R_e} M_g) \neq \{0\}$, then there exists an $r \in (N_g :_{R_e} M_g)$ such that $arm \neq 0$. Hence $0 \neq a(b+r)m \in N_g$. Since N_g is a weakly g-2-absorbing R_e-submodule of M_g, we have either $a(b+r) \in (N_g :_{R_e} M_g)$ or $am \in N_g$ or $(b+r)m \in N_g$, and hence either $ab \in (N_g :_{R_e} M_g)$ or $am \in N_g$ or $bm \in N_g$, which is impossible. Consequently, $am(N_g :_{R_e} M_g) = \{0\}$. With a same argument, we can show that $bm(N_g :_{R_e} M_g) = \{0\}$.

(iv) Assume that $(N_g :_{R_e} M_g)^2 m \neq \{0\}$, then there exist a_0, $b_0 \in (N_g :_{R_e} M_g)$ such that $a_0 b_0 m \neq 0$. By parts (i) and (iii), we have $0 \neq a_0 b_0 m = (a_0 + a)(b_0 + b)m \in N_g$. Since N_g is a weakly g-2-absorbing R_e-submodule of M_g, we have either $(a_0 + a)(b_0 + b) \in (N_g :_{R_e} M_g)$ or $(a_0 + a)m \in N_g$ or $(b_0 + b)m \in N_g$, and hence either $ab \in (N_g :_{R_e} M_g)$ or $am \in N_g$ or $bm \in N_g$, which is impossible. Consequently, $(N_g :_{R_e} M_g)^2 m = \{0\}$.

(v) Assume that $a(N_g :_{R_e} M_g)N_g \neq \{0\}$, then there exist $r_0 \in (N_g :_{R_e} M_g)$ and $x_0 \in N_g$ such that $ar_0x_0 \neq 0$. By parts (i), (ii), and (iii), we have $0 \neq ar_0x_0 = a(b + r_0)(m + x_0) \in N_g$. Since N_g is a weakly g-2-absorbing R_e-submodule of M_g, we have either $a(b + r_0) \in (N_g :_{R_e} M_g)$ or $a(m + x_0) \in N_g$ or $(b + r_0)(m + x_0) \in N_g$, and hence either $ab \in (N_g :_{R_e} M_g)$ or $am \in N_g$ or $bm \in N_g$, which is impossible. Consequently, $a(N_g :_{R_e} M_g)N_g = \{0\}$. With a same argument, we can show $b(N_g :_{R_e} M_g)N_g = \{0\}$. □

Theorem 3.6. *Let R be a G-graded ring, M be a graded R-module, N be a graded submodule of M and let $g \in G$ such that $(N_g :_{R_e} M_g)^2 N_g \neq \{0\}$. Then N_g is a weakly g-2-absorbing R_e-submodule of M_g, if and only if N_g is g-2-absorbing R_e-submodule of M_g.*

Proof. Suppose that N_g is a weakly g-2-absorbing R_e-submodule of M_g that is not a g-2-absorbing. Then, there exist $a, b \in R_e$ and $m \in M_g$, such that $abm \in N_g$, $am, bm \notin N_g$ and $ab \notin (N_g :_{R_e} M_g)$ Since $(N_g :_{R_e} M_g)^2 N_g \neq \{0\}$, there exist, $a_0, b_0 \in (N_g :_{R_e} M_g)$ and $x_0 \in N_g$ such that $a_0 b_0 x_0 \neq 0$. By Lemma 3.5, we have $a_0 b_0 x_0 = (a + a_0)(b + b_0)(m + x_0)$. Since N_g is a weakly g-2-absorbing R_e-submodule of M_g and $0 \neq a_0 b_0 x_0 = (a + a_0)(b + b_0)(m + x_0) \in N_g$, we conclude that either $(a + a_0)(m + x_0) \in N_g$ or $(b + b_0)(m + x_0) \in N_g$ or $(a + a_0)(b + b_0) \in (N_g :_{R_e} M_g)$ and hence either $ab \in (N_g :_{R_e} M_g)$ or $am \in N_g$ or $bm \in N_g$, a contradiction. Thus N_g is a g-2-absorbing R_e-submodule of M_g. The converse is obvious.

□

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